SUBJECT: MATHEMATICS

PAPER-II, UNIT-III

RADIAL AND TRANSVERSE ACCELERATIONS IN POLAR CURVE

SEMESTER-IV

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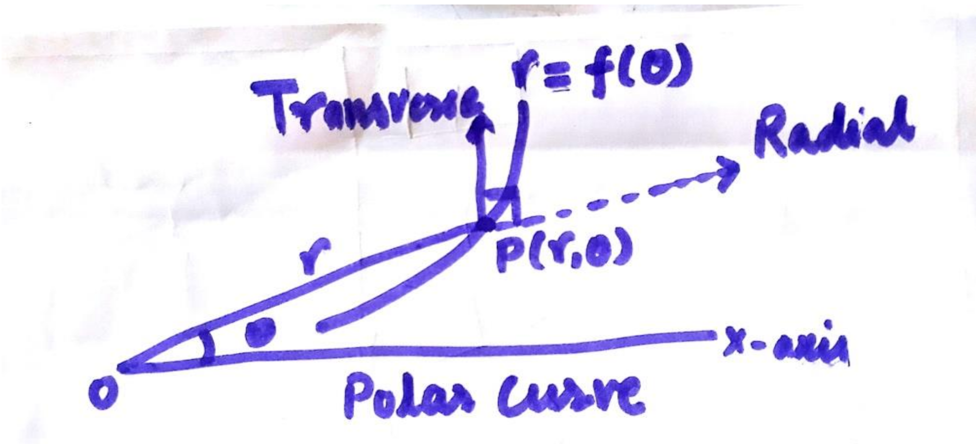
In this chapter we study components of velocities and accelerations in two mutually perpendicular directions in a Polar curve.

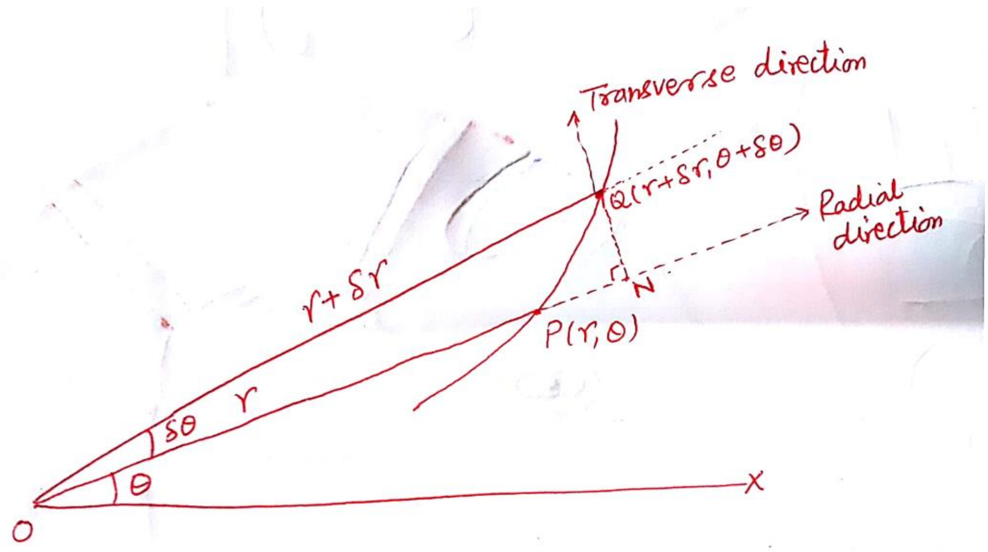
POLAR CURVE:

In a polar curve r = f(θ), there are two perpendicular directions, Radial and Transverse as shown in the figure.



VELOCITY AND ACCELERATION IN POLAR CO- ORDINATES (RADIAL AND TRANSVERSE VELOCITIES):

Consider a particle moves in a plane curve. Let P (r, θ) be its position in time t and $Q(r + \delta r, \theta + \delta\theta)$ be its position in time (t+ δt). Take 'O' as the pole and OX- as initial line. Velocity along the radius vector OP in the direction of r increasing is called the radial velocity and the velocity in the direction perpendicular to OP in the direction of Q increasing is called the transverse velocity.



So Radial Velocity at P = $\lim_{\delta t \to 0} \dfrac{displacement\ along\ OP\ in\ time\ \delta t}{\delta t}$

$$=\lim_{\delta t \to 0} \frac{PN}{\delta t} = \lim_{\delta t \to 0} \frac{ON - OP}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(r+\delta r)\cos\delta\theta - r}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(r+\delta r)\left[1 - \frac{(\delta\theta)^2}{2!} + \cdots\ldots\right] - r}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(r+\delta r)(1) - r}{\delta t}$$

$$(\text{neglecting higher powers of } \delta\theta)$$

$$= \lim_{\delta t \to 0} \frac{\delta r}{\delta t} = \frac{dr}{dt} = \dot{r}$$

So, Radial velocity = $\dot{r}$

Also, Transverse Velocity at P

$$= \lim_{\delta t \to 0} \frac{displacement\ perpendicular\ OP\ in\ time\ \delta t}{\delta t}$$

$$=\lim_{\delta t \to 0} \frac{QN}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(r+\delta r)\sin\delta\theta}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(r+\delta r)\left[\delta\theta - \frac{(\delta\theta)^3}{3!} + \cdots ....\right]}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(r+\delta r)(\delta\theta)}{\delta t}$$

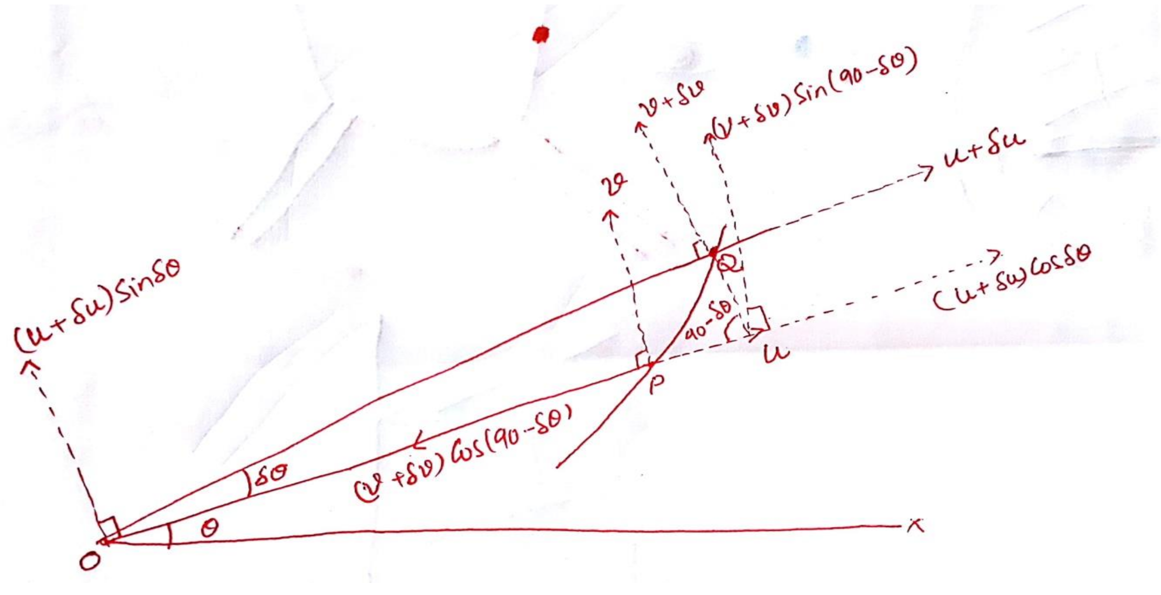
$\left(\text{neglecting higher powers of } \delta\theta\right)$

$$= \lim_{\delta t \to 0} r\frac{\delta\theta}{\delta t} = r\frac{d\theta}{dt} = r\dot{\theta}$$

So, Transverse velocity = $r\dot{\theta}$

RADIAL AND TRANSVERSE ACCELERATIONS:

Let u, v be the radial and transverse velocities at (r,θ) and ($u + \delta u$),(v +δv) radial and transverse velocities at Q ($r + \delta r$,$\theta + \delta\theta$).



So Radial acceleration at P = $\lim_{\delta t \to 0} \frac{\text{change in velocity along OP in time } \delta t}{\delta t}$

$$=\lim_{\delta t\to 0}\frac{(u+\delta u)\cos\delta\theta-(v+\delta v)\cos(90^{\circ}-\delta\theta)-u}{\delta t}$$

$$=\lim_{\delta t\to 0}\frac{(u+\delta u)\,(1)-(v+\delta v)\,(\delta\theta)-u}{\delta t}$$

$$=\lim_{\delta t\to 0}\frac{\delta u-v\delta\theta}{\delta t}$$

$$=\lim_{\delta t\to 0}\frac{\delta u}{\delta t}-\lim_{\delta t\to 0}v\frac{\delta\theta}{\delta t}$$

$$=\frac{du}{dt}-v\frac{d\theta}{dt}$$

$$=\frac{d}{dt}\left(\frac{dr}{dt}\right)-\left(r\frac{d\theta}{dt}\right)\frac{d\theta}{dt}\ \left(\text{as } u=\frac{dr}{dt}, v=r\frac{d\theta}{dt}\right)$$

$$So,\ \boldsymbol{Radial\ acceleration}=\boldsymbol{\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2}$$

Now, Transverse acceleration at P

$$=\lim_{\delta t\to 0}\frac{change\ in\ velocity\ perpendicular\ to\ OP\ in\ time\ \delta t}{\delta t}$$

$$=\lim_{\delta t\to 0}\frac{(u+\delta u)\sin\delta\theta+(v+\delta v)\sin(90^{\circ}-\delta\theta)-v}{\delta t}$$

$$=\lim_{\delta t\to 0}\frac{(u+\delta u)\,(\delta\theta)+(v+\delta v)\,(1)-v}{\delta t}$$

$$=\lim_{\delta t\to 0}\frac{u(\delta\theta)+\delta v}{\delta t}$$

$$=\lim_{\delta t\to 0}u\frac{\delta\theta}{\delta t}+\lim_{\delta t\to 0}\frac{\delta v}{\delta t}=u\frac{d\theta}{dt}+\frac{dv}{dt}$$

= $\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right)+\frac{d}{dt}\left(r\,\frac{d\theta}{dt}\right)$ (as u= $\frac{dr}{dt}$,v=r $\frac{d\theta}{dt}$)

$$= \quad 2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)$$

So, Transverse **acceleration =** $\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)$

	Velocities and Accelerations	Magnitude
1	Radial Component of velocity	$\dot{\boldsymbol{r}}$
2	Transverse Component of velocity	$\boldsymbol{r}\dot{\boldsymbol{\theta}}$
3	Radial component of acceleration	$\ddot{\boldsymbol{r}} - \boldsymbol{r}(\dot{\boldsymbol{\theta}})^2$
4	Transverse component of acceleration	$\frac{\mathbf{1}}{\boldsymbol{r}}\frac{\boldsymbol{d}}{\boldsymbol{dt}}\left(\boldsymbol{r}^2\dot{\boldsymbol{\theta}}\right)$

Example: If a point moves so that its radial velocity is k times its transverse velocity, then show that its path is an equiangular spiral.

Solution:

Given, radial velocity = k × transverse velocity

$\Rightarrow \dot{r} = kr\dot{\theta}$

i.e. $\frac{dr}{dt} = kr\frac{d\theta}{dt} \Longrightarrow \frac{dr}{r} = kd\theta$

Integrating, log r = k θ + log A, A = constant

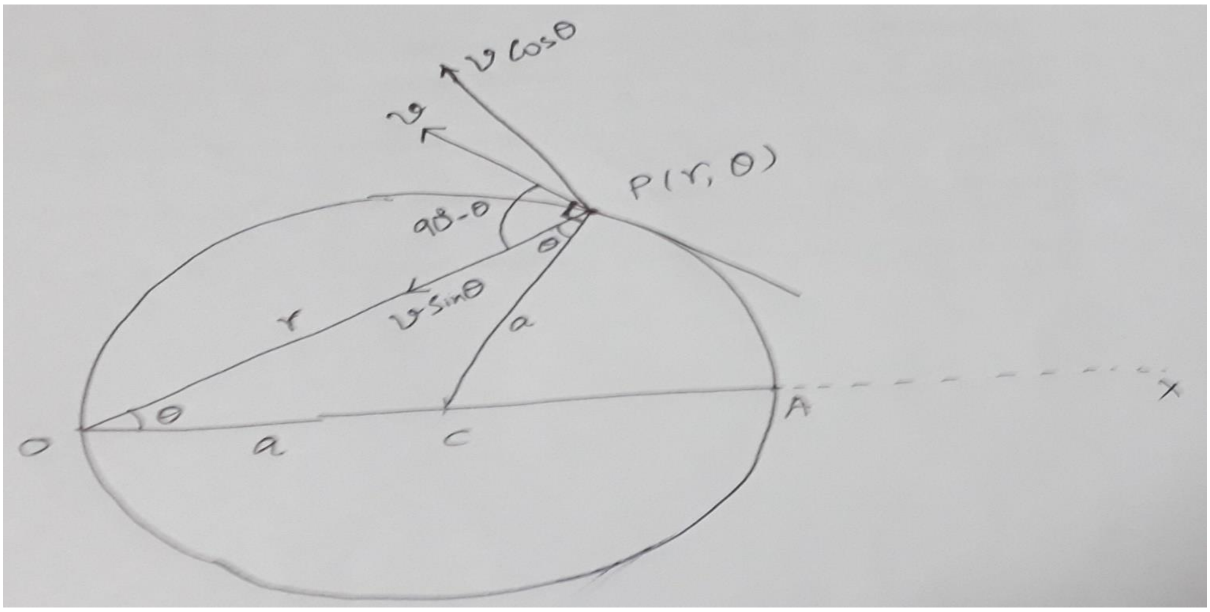
i.e. r = A $e^{k\theta}$

which is an equiangular spiral.

Example 2

A point moves in a circular path of radius "a" so that its angular velocity about a fixed point in the circumference of the circle is constant, equal to w .Show that the resultant acceleration of the point at every point of the path is of constant magnitude 4 aw^2 .

Solution: Let 'O' be the fixed point (pole), OC – initial line. Polar equation of the circle is r = 2 a cos θ . Let P (r, θ) be the position at time 't' then angular velocity about O is $\dot{\theta}$ = w (constant)



Radial velocity = $\dot{r}$ = -(2a sin θ) $\dot{\theta}$ = -2aw sin θ

$\Rightarrow \ddot{r}$ = -(2aw cos θ)$\dot{\theta}$ = -2aw^2 cos θ

= -w^2 (2a cos θ)

= -$w^2.r$

Radial acceleration at P =$\ddot{r} - r(\dot{\theta})^2$

= -$w^2.r$-$w^2.r$

= - 2w^2r = -2w^2 (2a cos θ)

= - 4aw^2 cos θ

Transverse acceleration at P = $\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = \frac{1}{r}$ $w.2r\,\dot{r}$

= 2 w (- 2aw sinθ) = - 4a sinθ

So, Resultant acceleration =$\sqrt{\left(-4a\,w\,2\cos\theta\right)^2 + \left(-4aw\,2\sin\theta\right)^2}$

=4aw ^{2}